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Long-time asymptotics of the mean-field magnetohydrodynamics equation

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Abstract
It is shown that a magnetic field satisfying the mean-field magnetohydrodynamics equation with zero mean velocity and without energy input from the outside possesses a Lyapunov function, which is a combination of the magnetic energy and the helicity. As a consequence, if the mean magnetic field remains uniformly bounded for all time, the field tends asymptotically in time to an attractor formed by force-free states.

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1. Introduction

Under the magnetohydrodynamic approximation, the magnetic field $\boldsymbol{B}$ in an incompressible plasma of velocity $\boldsymbol{u}$, viscosity υ, resistivity η and kinetic pressure p satisfies in the absence of forcing the equations

$$\begin{aligned}\frac{\partial \boldsymbol{u}}{\partial t} &= \nu\Delta\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \boldsymbol{J}\times\boldsymbol{B} - \nabla p \\ \frac{\partial \boldsymbol{B}}{\partial t} &= \eta\Delta\boldsymbol{B} + \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) \\ \nabla\cdot\boldsymbol{u} &= \nabla\cdot\boldsymbol{B} = 0\end{aligned} \tag{1}$$

where $\boldsymbol{J} = \nabla\times\boldsymbol{B}$ is the current density.

With the usual homogeneous boundary conditions in the domain Ω, which imply no input of energy from the outside, the energy E of any trajectory $t \to (\boldsymbol{u}(t), \boldsymbol{B}(t))$ is decreasing in time:

$$\begin{aligned}E(t) &= \tfrac{1}{2}\int_\Omega u(t)^2 + B(t)^2\,\mathrm{d}V \\ \dot{E}(t) &= -\int_\Omega \nu\omega(t)^2 + \eta J(t)^2\,\mathrm{d}V\end{aligned} \tag{2}$$

where $\omega = \nabla \times \boldsymbol{u}$ indicates the vorticity of the flow. This implies that both velocity and magnetic field will eventually decay to zero. Another important function is the *magnetic helicity*,

$$H = \frac{1}{2} \int_{\Omega} \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V \tag{3}$$

where $\boldsymbol{A}$ is a vector potential for $\boldsymbol{B}$ ($\nabla \times \boldsymbol{A} = \boldsymbol{B}$). This is invariant for ideal plasmas ($v = \eta = 0$). In general, however,

$$\dot{H} = -\eta \int_{\Omega} \boldsymbol{J} \cdot \boldsymbol{B} \, \mathrm{d}V. \tag{4}$$

In principle there is no mathematical reason why H should decay more slowly than E. However, there are good physical arguments to believe that this is the case in MHD turbulence (see e.g. [1] and references therein). If we take H as constant (which it is not, although as asserted it is likely to vary more slowly than the energy) and minimize E under this constraint, one finds a force-free state

$$\boldsymbol{J} = \mu \boldsymbol{B}. \tag{5}$$

In fact the assumed quasi-invariance of H is the main example of an inverse cascade in three-dimensional MHD, and the appearance of large-scale features in the magnetic field is a consequence of it. We intend to provide a partial analogue of (5) for the mean-field MHD equation. This is one of the main tools in the study of astrophysical turbulent plasmas. The equation is as follows:

$$\frac{\partial \bar{\boldsymbol{B}}}{\partial t} = \nabla \times \left(-(\eta + \beta) \nabla \times \bar{\boldsymbol{B}} + \bar{\boldsymbol{u}} \times \bar{\boldsymbol{B}} + \alpha \bar{\boldsymbol{B}} \right). \tag{6}$$

Here $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{B}}$ are means of the velocity and the magnetic field, β a turbulent diffusivity and α an important term representing the possible enhancement of the field by fluctuating small-scale motions. For the classic derivation of this equation, see [2]; for the first-order smoothing approach, [3]; for an excellent critical study of it, [4]. Mathematical questions related to it appear in [5]. Applications of this equation to specific astrophysical dynamo models are countless, and many of them rather successful. This fact gives credibility to (6) and makes it worthwhile to undertake a study of possible Lyapunov functions for its associated trajectories, in order to ascertain where such trajectories are likely to end, i.e. to study the long-term asymptotics of mean fields satisfying (6). Such a study would depend heavily on the behaviour of $\bar{\boldsymbol{u}}$, which, in contrast to the full MHD system (1), must be taken as a datum. Strictly speaking (6) is a refinement of the induction equation for the magnetic field and not of the full system. The induction equation, by itself, has no Lyapunov function unless the velocity is zero, where it reduces to a simple diffusion equation. Here the situation is different: we may assume that the mean velocity is zero (or constant, so that it would vanish after a Galilean change of variables) and the alpha term, incorporating the effect of fluctuating velocities, may still enhance the field to prevent decay. Hence we assume $\bar{\boldsymbol{u}} = \boldsymbol{0}$; from now on we will drop the bar in $\bar{\boldsymbol{B}}$. As for the α and β terms, their expressions are not clear *a priori*, but, since turbulence is suppressed by large mean magnetic fields, they certainly must decay with the size of $\boldsymbol{B}$ (alpha-quenching). An expression for them [6] is

$$\begin{aligned} \alpha &= \frac{\alpha_0 f}{1 + kB^2} \\ \beta &= \frac{\beta_0 f}{1 + kB^2} \end{aligned} \tag{7}$$

where f is a point function (a multiple of the cosinus of the latitude angle for axisymmetric problems), and k a positive constant. The usual situation is that the turbulence adds to the diffusivity, i.e. that β is positive. Hence we will assume that f is positive and $\beta_0 > 0$. We will take $\alpha_0 > 0$; if $\alpha_0 < 0$, the same results hold by using instead the opposite Lyapunov function. The size of k determines the effect of the alpha and beta terms for larger fields. There is some controversy about it [7,8] but this will not affect our study. Finally, since β is much larger than η for moderate-sized turbulent fields, which are the ones we wish to analyse, we will drop the term in η from equation (6).

Our main conclusions will be as follows: for boundary conditions for which there is no input of Lorentz force from the outside, there exists a Lyapunov function L which makes the states with minimal L force-free ones. However, the constant μ of (5) must now be precisely α_0/β_0, which imposes additional constraints upon them. Also, it is not clear that any trajectory should tend to any of these states (as is often assumed), but we prove that the set of these force-free fields is an attractor for any trajectory where the magnetic field remains uniformly bounded.

2. The Lyapunov function

In addition to equation (6), the magnetic field must satisfy some initial and boundary conditions. These determine the space of functions where we work; all of them are subspaces of the set of square-integrable functions $L^2(\Omega)$. Thus, for periodic conditions, we set $\Omega = [0, 1]^3$, and

$$\mathcal{H} = \left\{ \boldsymbol{B} \in L^2(\Omega)^3 / \nabla \cdot \boldsymbol{B} = 0,\ \boldsymbol{B} \cdot \boldsymbol{n}|_{\partial\Omega} \text{ periodic}, \int_\Omega \boldsymbol{B}\,\mathrm{d}V = \boldsymbol{0} \right\}. \tag{8}$$

The condition $\nabla \cdot \boldsymbol{B} = 0$ must be understood in the sense of distributions. For these solenoidal fields the trace $\boldsymbol{B} \cdot \boldsymbol{n}$ makes sense at the boundaries. For other cases, one simply sets

$$\mathcal{H} = \{ \boldsymbol{B} \in L^2(\Omega)^3 / \nabla \cdot \boldsymbol{B} = 0,\ \boldsymbol{B} \cdot \boldsymbol{n}|_{\partial\Omega} = 0 \}. \tag{9}$$

We could choose additional boundary conditions, such as Dirichlet ones ($\boldsymbol{B}|_{\partial\Omega} = \boldsymbol{0}$), or perfect conductor ones ($\boldsymbol{J} \times \boldsymbol{n}\ |_{\partial\Omega} = \boldsymbol{0}$), but it is simpler to remain within $\mathcal{H}$ and impose the necessary conditions upon every trajectory. With such generality we cannot prove results of existence and regularity for all time. Therefore we simply assume that the solution to (6) with an initial condition $\boldsymbol{B}(0) = \boldsymbol{B}_0$ exists for all time, and is smooth enough for the current density $\boldsymbol{J} = \nabla \times \boldsymbol{B}$ to be square integrable in the domain Ω.

Our next hypothesis is that there are no inputs of energy from the outside of the domain. We will see that this condition means that the inequality

$$\int_{\partial\Omega} \beta (\boldsymbol{J} \times \boldsymbol{B}) \cdot \boldsymbol{n}\,\mathrm{d}\sigma \geqslant 0 \tag{10}$$

holds for all time. Although it would be conceptually simpler to assume that there is no exchange of energy (i.e. that the inequality (10) is an equality) we will try to be as general as possible. Also, it makes sense that if one allows magnetic energy to flow into the domain one may obtain any evolution one wishes, whereas by allowing energy only to escape the system is left to its own resources and the study of its limit states is possible. Notice that this integral vanishes for the periodic case (because the integrand itself is periodic, and $\boldsymbol{n}$ has opposite signs on opposite sides of the box), for Dirichlet conditions and for the perfect conductor case. Analogously to (2), the magnetic energy is defined by

$$E = \tfrac{1}{2} \int_\Omega B^2\,\mathrm{d}V \tag{11}$$

and the magnetic helicity is defined by (3). Note that if we assume that the domain Ω is simply connected, any two vector potentials for $\boldsymbol{B}$ differ in a gradient, $\nabla\Phi$. Since always

$$\int_\Omega \boldsymbol{B}\cdot\nabla\Phi\,\mathrm{d}V = \int_{\partial\Omega}\Phi\boldsymbol{B}\cdot\boldsymbol{n}\,\mathrm{d}\sigma = 0 \tag{12}$$

the election of $\boldsymbol{A}$ does not affect the value of H. We choose therefore $\boldsymbol{A}$ with $\boldsymbol{A}\times\boldsymbol{n}|_{\partial\Omega} = 0$, an election which satisfies a bound of the type

$$\|\boldsymbol{A}\|_2 \leqslant M\|\boldsymbol{B}\|_2 \tag{13}$$

for some constant M depending on Ω [9]. Therefore

$$|H| \leqslant \tfrac{1}{2}M\|\boldsymbol{B}\|_2^2 = ME. \tag{14}$$

Consider the evolution of the magnetic energy of a trajectory $\boldsymbol{B}(t)$. We have

$$\begin{aligned}\dot{E} = \int_\Omega \boldsymbol{B}\cdot\dot{\boldsymbol{B}}\,\mathrm{d}V &= \int_\Omega \boldsymbol{B}\cdot\nabla\times(-\beta\boldsymbol{J}+\alpha\boldsymbol{B})\,\mathrm{d}V\\ &= -\int_\Omega \nabla\cdot(\boldsymbol{B}\times(-\beta\boldsymbol{J}+\alpha\boldsymbol{B}))\,\mathrm{d}V + \int_\Omega(-\beta\boldsymbol{J}+\alpha\boldsymbol{B})\cdot\boldsymbol{J}\,\mathrm{d}V\\ &= -\int_{\partial\Omega}\beta(\boldsymbol{J}\times\boldsymbol{B})\cdot\boldsymbol{n}\,\mathrm{d}\sigma + \int_\Omega -\beta J^2 + \alpha\boldsymbol{B}\cdot\boldsymbol{J}\,\mathrm{d}V\\ &\leqslant \int_\Omega -\beta J^2 + \alpha\boldsymbol{B}\cdot\boldsymbol{J}\,\mathrm{d}V.\end{aligned} \tag{15}$$

As for the evolution of the magnetic helicity $H(t)$, one obtains

$$\dot{H} = \tfrac{1}{2}\int_\Omega \dot{\boldsymbol{A}}\cdot\boldsymbol{B} + \boldsymbol{A}\cdot\dot{\boldsymbol{B}}\,\mathrm{d}V. \tag{16}$$

Now, since Ω is simply connected, $\boldsymbol{A}$ satisfies the ‘uncurled’ equation

$$\frac{\partial\boldsymbol{A}}{\partial t} = -\beta\boldsymbol{J} + \alpha\boldsymbol{B} + \nabla\Psi \tag{17}$$

for some time-dependent potential Ψ. Since

$$\int_\Omega \boldsymbol{B}\cdot\nabla\Psi\,\mathrm{d}V = 0$$

we are left with

$$\begin{aligned}\dot{H} = \tfrac{1}{2}\int_\Omega &-\beta\boldsymbol{J}\cdot\boldsymbol{B} + \alpha B^2 + \boldsymbol{A}\cdot\nabla\times(-\beta\boldsymbol{J}+\alpha\boldsymbol{B})\,\mathrm{d}V\\ &= \tfrac{1}{2}\int_\Omega -\beta\boldsymbol{J}\cdot\boldsymbol{B} + \alpha B^2 - \nabla\cdot(\boldsymbol{A}\times(-\beta\boldsymbol{J}+\alpha\boldsymbol{B})) + \boldsymbol{B}\cdot(-\beta\boldsymbol{J}+\alpha\boldsymbol{B})\,\mathrm{d}V\\ &= \int_\Omega -\beta\boldsymbol{J}\cdot\boldsymbol{B} + \alpha B^2\,\mathrm{d}V - \tfrac{1}{2}\int_{\partial\Omega}(\boldsymbol{A}\times(-\beta\boldsymbol{J}+\alpha\boldsymbol{B}))\cdot\boldsymbol{n}\,\mathrm{d}\sigma.\end{aligned} \tag{18}$$

Since $\boldsymbol{A}\times\boldsymbol{n} = \boldsymbol{0}$ at $\partial\Omega$, the boundary integral vanishes. Let $\lambda = \beta/\alpha = \beta_0/\alpha_0$. Then

$$(\lambda E - H)^{\cdot} \leqslant \int_\Omega -\alpha\lambda^2 J^2 + 2\alpha\lambda\boldsymbol{B}\cdot\boldsymbol{J} - \alpha B^2\,\mathrm{d}V = -\int_\Omega \alpha|\lambda\boldsymbol{J} - \boldsymbol{B}|^2\,\mathrm{d}V. \tag{19}$$

Since $\alpha > 0$, $\lambda E - H$ decreases in time. It may only become stationary if $\lambda\boldsymbol{J} = \boldsymbol{B}$, in which case obviously $\dot{\boldsymbol{B}}(t) = \boldsymbol{0}$ and the trajectory itself is stationary. To be precise, the identity above should hold almost everywhere (i.e. in $\mathcal{H}$). We will see in the next section that any such force-free state is necessarily smooth and therefore the identity holds everywhere.

Notice that, in general, the system is in equilibrium if $-\beta\boldsymbol{J}+\alpha\boldsymbol{B} = \nabla\Phi$ for some potential Φ. If moreover $\boldsymbol{B}$ satisfies our boundary conditions, $(\lambda E - H)^{\cdot} = 0$ and necessarily

$$0 \leqslant -\int_\Omega \alpha|\lambda\boldsymbol{J} - \boldsymbol{B}|^2\,\mathrm{d}V = -\int_\Omega \frac{1}{\alpha}|\nabla\Phi|^2\,\mathrm{d}V \leqslant 0.$$

The only possibility is therefore $\nabla\Phi = \mathbf{0}$, i.e. $\lambda\boldsymbol{J} = \boldsymbol{B}$.

Thus $\lambda E - H$ is a Lyapunov function for our trajectory. It always decreases unless $\lambda\boldsymbol{J} = \boldsymbol{B}$, which is a very particular force-free state: the constant of proportionality must be precisely the ratio between α and β. How likely is this? Notice that in this case, also $\lambda\nabla\times\boldsymbol{J} = -\lambda\Delta\boldsymbol{B} = (1/\lambda)\boldsymbol{B}$. Hence $-\Delta\boldsymbol{B} = (1/\lambda^2)\boldsymbol{B}$. If $\boldsymbol{B}$ satisfies one of the classical boundary conditions defined before, which make the Laplacian an elliptic, dissipative operator with a sequence of eigenvalues tending to $-\infty$, $-1/\lambda^2$ should be one of them. For the ratio $(\beta/\alpha)^2$, which is an *a priori* quantity, to coincide with one of a discrete set of parameters characteristic of the domain Ω seems rather unlikely, although it may happen [6]. It is for precisely this reason that we have chosen the most general condition (10), which does not determine the spectrum of Δ and therefore could allow a continuum of solutions.

If we know *a priori* that the energy of the trajectory remains bounded, $E \leqslant C$, as we will assume in the next section, by (14) $|\lambda E - H| \leqslant (\lambda + M)C$, and the Lyapunov function is bounded. This demands $(\lambda E - H)^{\cdot} \to 0$ as $t \to \infty$, and $\lambda E - H$ decreases to a certain real value. This does not need to imply that $\boldsymbol{B}(t)$ tends to a particular force-free state, because these are not isolated points in the phase space. In fact they form a linear space (with any of the above boundary conditions, a finite-dimensional one). Many authors tend to think that a system tends to relax to a single minimum energy (or minimum Lyapunov function) state [10], but this is something that must be proved in every case.

3. Force-free states as attractors

In this section we will make use of some theorems and notations of functional analysis, concerning the Sobolev space $H^1(\Omega)$ of functions whose differential is square integrable and the fact that any ball in a Hilbert space is *weakly compact*. We will use the fact [9] that $H^1(\Omega)^3$ coincides with the space of functions $\boldsymbol{w}$ of $L^2(\Omega)^3$ such that both $\nabla\times\boldsymbol{w}$ and $\nabla\cdot\boldsymbol{w}$ are square integrable, and $\boldsymbol{w}\cdot\boldsymbol{n}$ lies within a certain space $H^{1/2}(\partial\Omega)$ of functions at the boundary. In our case, this means that any function $\boldsymbol{B}$ within $\mathcal{H}$ such that its curl is square integrable lies within $H^1(\Omega)^3$, and its H^1-norm

$$\|\boldsymbol{B}\|^2_{H^1} = \|\boldsymbol{B}\|^2_2 + \|\nabla\boldsymbol{B}\|^2_2$$

is equivalent to the L^2-norm of the current,

$$\|\boldsymbol{J}\|^2_2 = \int_\Omega |\nabla\times\boldsymbol{B}|^2\,\mathrm{d}V.$$

We intend to prove that the set $\mathcal{A} = \{\boldsymbol{B}\in\mathcal{H} : \lambda\boldsymbol{J} = \boldsymbol{B}\}$ of force-free states attracts every trajectory within $\mathcal{H}$ which remains uniformly bounded for all time.

First let us look at the set $\mathcal{A}$. Although in principle the value of $\boldsymbol{J} = \nabla\times\boldsymbol{B}$ must be understood in the sense of distributions, if $\lambda\boldsymbol{J} = \boldsymbol{B}$ then $\boldsymbol{J}\in\mathcal{H}$. As stated before, this implies $\boldsymbol{B}\in H^1(\Omega)^3\cap\mathcal{H}$. Since $\boldsymbol{J}\in\mathcal{H}$ and $\lambda\nabla\times\boldsymbol{J} = \boldsymbol{J}$, $\boldsymbol{J}$ itself belongs to $H^1(\Omega)^3\cap\mathcal{H}$. This shows that $\boldsymbol{B}\in H^2(\Omega)^3\cap\mathcal{H}$ [9], where H^2 is the space of square-integrable functions whose differential belongs to H^1. We may iterate this process and find that any $\boldsymbol{B}\in\mathcal{A}$ is indefinitely differentiable.

However, for our purposes it is sufficient to know $\mathcal{A}\subset H^1(\Omega)^3$. Since the embedding $H^1(\Omega)^3 \to L^2(\Omega)^3$ is compact, and for $\boldsymbol{B}\in\mathcal{A}$ we have $\|\boldsymbol{J}\|_2 \leqslant (1/\lambda)\|\boldsymbol{B}\|_2$, we find that for any closed ball $\bar{B}(0, R)$ in $\mathcal{H}$ the set $\mathcal{A} \cap \bar{B}(0, R)$ is compact in $\mathcal{H}$, i.e. every bounded set within $\mathcal{A}$ is relatively compact.

Now let $t \to \boldsymbol{B}(t)$ be any trajectory bounded for all time. We intend to prove that its ω-limit is contained within $\mathcal{A}$. Were it not so, there would be a sequence $t_n \to \infty$ and some $r > 0$ such that for all n the distance within the space $\mathcal{H}$

$$d(\boldsymbol{B}(t_n), \mathcal{A}) \geqslant r. \tag{20}$$

Since $\|\boldsymbol{B}(t_n)\|_\infty$ is bounded, so is $\|\boldsymbol{B}\|_2$; as stated in the previous section, the function $\lambda E - H$ is bounded for all time for this trajectory, tends to some value when $t \to \infty$ and $(\lambda E - H)^{\cdot}(t)$ tends to zero. In particular, from some time t_0 on,

$$(\lambda E - H)^{\cdot} = \int_\Omega \alpha |\lambda \boldsymbol{J} - \boldsymbol{B}|^2 \,\mathrm{d}V \leqslant 1. \tag{21}$$

Since (say) $\|\boldsymbol{B}(t)\|_\infty \leqslant N$, $\alpha \geqslant \alpha_0/(1 + kN^2)$. (This is the only time where the hypothesis of the uniform boundedness of $\boldsymbol{B}(t)$ is used.) This means that for $t \geqslant t_0$,

$$\|\boldsymbol{J}(t)\|_2 \leqslant \frac{1}{\lambda}\|\boldsymbol{B}(t)\|_2 + \frac{1}{\lambda}\left(\frac{1 + kN^2}{\alpha_0}\right)^{\frac{1}{2}}. \tag{22}$$

Thus the set $\{\boldsymbol{J}(t) : t \geqslant t_0\}$ is bounded in $L^2(\Omega)$; as stated above, this means that $\{\boldsymbol{B}(t) : t \geqslant t_0\}$ is bounded in $H^1(\Omega)^3 \cap \mathcal{H}$, and therefore relatively compact in $\mathcal{H}$. We now use the fact that any closed ball in $H^1(\Omega)^3 \cap \mathcal{H}$ (or in $\mathcal{H}$) is compact and metrizable with the weak topology: i.e. there exist $\boldsymbol{J}_0 \in \mathcal{H}$, $\boldsymbol{B}_0 \in H^1(\Omega)^3 \cap \mathcal{H}$ and a subsequence of (t_n), denoted again by (t_n), such that

$$\begin{aligned} &\int_\Omega (\boldsymbol{J}(t_n) - \boldsymbol{J}_0)\boldsymbol{\Phi} \,\mathrm{d}V \to 0 \qquad \forall \boldsymbol{\Phi} \in \mathcal{H} \\ &\int_\Omega (\boldsymbol{B}(t_n) - \boldsymbol{B}_0)\boldsymbol{\Phi} \,\mathrm{d}V \to 0 \qquad \forall \boldsymbol{\Phi} \in \mathcal{H}. \end{aligned} \tag{23}$$

As a matter of fact we could take a wider class of Φ for the second convergence, but this is unnecessary. Since the set $\{\boldsymbol{B}(t) : t \geqslant t_0\}$ is relatively compact in $\mathcal{H}$, we may assume that $\boldsymbol{B}(t_n) \to \boldsymbol{B}_0$ also in the topology of $\mathcal{H}$.

We must prove that $\boldsymbol{B}_0 \in \mathcal{A}$. Since $\|\lambda \boldsymbol{J}(t_n) - \boldsymbol{B}(t_n)\|_2 \to 0$, there exists a subsequence of the integrand converging to zero almost everywhere. The same may be said of the convergence $\boldsymbol{B}(t_n) \to \boldsymbol{B}_0$. For this subsequence (again denoted by (t_n)), $\boldsymbol{J}(t_n)$ must tend almost everywhere to some function $\boldsymbol{J}_1$. By using the convergence theorems and (25), we find that $\boldsymbol{J}_0 = \boldsymbol{J}_1$ a.e., so $\lambda \boldsymbol{J}_0 = \boldsymbol{B}_0$ in $\mathcal{H}$. By our previous argument, $\boldsymbol{B}_0 \in \mathcal{A}$ and it is a smooth field. Hence we have found a subsequence of the original one tending to an element of $\mathcal{A}$, which contradicts (22).

It is difficult to provide a physically relevant case where the precise situation described above occurs in all certainty. However, in [11] a model of isotropic helical turbulent magnetic field where α and β satisfy a relation of proportionality such as (7) is presented. The resulting numerical evolution produces nearly force-free states. Although apparently the field with any initial condition tends to one of these states, very small variations of these initial conditions yield a different limit. Given the inherent difficulties of long-term numerical evolution, this behaviour could indicate that the magnetic field tends to a robust attractor formed by force-free states, as described in our result. Attraction to a single state does not appear to be robust.

4. Conclusions

A magnetic field satisfying the mean-field magnetohydrodynamics equation with zero mean velocity, and whose alpha and beta terms are proportional, under rather general boundary conditions possesses a Lyapunov function which is a combination of the magnetic energy and helicity. This function is strictly decreasing for any non-stationary trajectory, and its only equilibrium points are certain force-free fields. While it is not clear that any trajectory should tend to any of these steady states, the set of them is an attractor for any uniformly bounded trajectory.

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